

# Walking in the SU(N)

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We study the phase diagram as function of the number of colours and flavours of asymptotically free non-supersymmetric theories with matter in higher dimensional representations of arbitrary SU(N) gauge groups. Since matter in higher dimensional representations screens more than in the fundamental a general feature is that a lower number of flavours is needed to achieve a near-conformal theory. We study the spectrum of the theories near the fixed point and consider possible applications of our analysis to the dynamical breaking of the electroweak symmetry.

## I. INTRODUCTION

In this article we portray the phase diagram, as a function of the number of colours and the number of flavours, of non-supersymmetric gauge field theories with matter in higher dimensional representations of SU(N) gauge groups. Strongly interacting non-Abelian theories generally feature a coupling constant which varies with the energy scale; it *runs*. This is caused by the antiscreening due to the charged gauge bosons (gluons). This antiscreening is competing with the screening contribution from matter fields. There is another feature of strongly interacting theories which comes into play in this context, that is chiral symmetry breaking. This formation of a  $\langle\bar{\psi}\psi\rangle$  condensate below the chiral symmetry breaking scale renders the fermions massive and results in their decoupling from the dynamics. In that case the antiscreening of the gluons becomes more important again. With sufficient matter content an infrared fixed point of the coupling constant can be reached, before chiral symmetry breaking is triggered, that is there exists a conformal phase. For a number of flavours slightly below the value for which the conformal phase is present, the coupling constant is evolving slowly. It stays almost constant over a range of energy scales. One says it *walks* instead of *runs*. These features will be discussed in more detail below.

We are particularly interested in these walking theories: It is possible that, besides Quantum Chromodynamics (QCD), new strongly interacting theories will emerge when exploring the unknown territory beyond the Standard Model (SM) of particle interactions. For example, to avoid unnaturally large quantum corrections to the mass scale of the electroweak theory arising in the Higgs sector of the SM one can replace the elementary Higgs by a strongly coupled sector. This approach has been named technicolour [1]. The generation of the masses of the standard model fermions requires extended technicolour interactions, which are consistent with the observed amount of flavour-changing neutral currents and lepton number violation for technicolour theories pos-

sessing a sufficient amount of walking [2, 3, 4, 5, 6]. The simplest of such models which also passes the electroweak precision tests (like, for example the experimental bounds on the oblique parameters) requires fermions in higher dimensional representations of the technicolour gauge group [7, 8, 9, 10]. Matter in a higher dimensional representation screens more strongly than matter in the fundamental representation, whence a smaller number of flavours is required in order to achieve a given amount of screening. In the recent past we have studied the gauge dynamics of fermions transforming according to the two-index representations of SU(N). Here we extend the analysis to a generic asymptotically free gauge theory with fermions in various higher dimensional representations of the SU(N) group. Some general structures are unveiled.

After a general and fairly complete analysis (Sect. II) we explore the various ways these theories can be used to break the electroweak symmetry (Sect. III). Among other things we compute the left-right vector correlator in perturbation theory. Thanks to the results found in [11, 12, 13] and corroborated more recently by AdS/QCD-based computations [14] we know that near the conformal window the perturbative value of the oblique parameter  $S$  provides a conservative estimate, that is an upper bound. Finally, using the four-dimensional renormalisation group approach we explore the spectrum near the fixed point (Sect. IV). In Sect. V, we summarise the results.

## II. PHASE DIAGRAM

### A. Review of the basic tools

We start our journey by reviewing the two-loop  $\beta$  function for a generic non-Abelian gauge theory with fermionic matter in a given representation R of

$SU(N)[15]^1$ :

$$\beta(g) = -\beta_0 \frac{g^3}{(4\pi)^2} - \beta_1 \frac{g^5}{(4\pi)^4}, \quad (1)$$

$$2N\beta_0 = \frac{11}{3}C_2(G) - \frac{4}{3}T(R), \quad (2)$$

$$(2N)^2\beta_1 = \frac{34}{3}C_2^2(G) - \frac{20}{3}C_2(G)T(R) - 4C_2(R)T(R). \quad (3)$$

$C_2(R)$  stands for the quadratic Casimir operator of the representation  $R$ ,

$$2NX_R^a X_R^a = C_2(R)\mathbb{1}, \quad (4)$$

where  $X_R^a$  are the generators in the representation  $R$ .  $C_2(G)$  is the quadratic Casimir operator of the adjoint representation.  $T(R)$  is the trace normalisation factor for the representation  $R$ .  $T(R)$  is connected to the quadratic Casimir operator,  $C_2(R)$ , by [16]

$$N_f C_2(R) d(R) = T(R) d(G), \quad (5)$$

where  $d(R)$  denotes the dimension of the representation,  $R$ , and, accordingly,  $d(G)$  the dimension of the adjoint representation  $G$  that is the number of generators.  $N_f$  stands for the number of flavours.

A theory's loss of asymptotic freedom shows itself in a change of sign of the first coefficient,  $\beta_0$ , of the  $\beta$  function. Hence, the number of flavours,  $N_f^I[R]$ , above which the loss occurs satisfies  $\beta_0[N_f^I(R)] \stackrel{!}{=} 0$ . Therefore, from Eqs. (2) and (5) we obtain

$$N_f^I[R] := \frac{11}{4} \frac{d(G)C_2(G)}{d(R)C_2(R)}, \quad (6)$$

which is thus proportional to the ratio of the second order indices,  $d(R)C_2(R)$ , of the adjoint representation  $G$  and the representation  $R$ , respectively. For matter in the adjoint representation,  $R = G$ , the number of flavours above which asymptotic freedom is lost is independent of the number of colours and equal to  $\frac{11}{4}$ .

Already for a smaller number of flavours than the one at which asymptotic freedom is lost, the theory develops a Banks-Zaks infrared fixed point [17]. It appears as soon as the second coefficient of the  $\beta$  function,  $\beta_1$ , changes sign, that is at  $\beta_1[N_f^{III}(R)] \stackrel{!}{=} 0$ . Eqs. (3) and (5) then lead to

$$N_f^{III}[R] = \frac{d(G)C_2(G)}{d(R)C_2(R)} \frac{17C_2(G)}{10C_2(G) + 6C_2(R)}. \quad (7)$$

This fixed point will not come into play if, before it is reached, the coupling constant  $\alpha$  becomes so large that chiral symmetry breaking is triggered.

In this context, in order to determine the number of flavours  $N_f^{II}[R]$  above which the theory becomes conformal, we employ the criterion derived in [18, 19] and follow the discussion in [7]: Whether chiral symmetry can be broken depends on the relative order (with respect to the energy scale) of the values of the coupling constant at which a conformal fixed point is encountered,  $\alpha_*$ , and where chiral symmetry breaks,  $\alpha_c$ . When following the renormalisation group flow from higher to lower energies the coupling  $\alpha$  keeps growing as long as the  $\beta$  function remains finite and negative. If the value  $\alpha_c$  is reached, the fermions decouple, their screening effect is lost, and only the antiscreening of the gauge bosons remains. In that case the conformal fixed point cannot be reached. If, on the other hand,  $\alpha_*$  is met before chiral symmetry is broken, the coupling freezes (because the  $\beta$  function becomes zero). Now, in turn, the value  $\alpha_c$  of the coupling required for chiral symmetry breaking cannot be attained.

In [18, 19], the critical value  $\alpha_c$  of the coupling constant for which chiral symmetry breaking occurs is defined as the value for which the anomalous dimension of the quark mass operator becomes unity,  $\gamma \stackrel{!}{=} 1$ . According to [18], in ladder approximation the critical value of the coupling is given by <sup>2</sup>

$$\alpha_c = \frac{2\pi N}{3C_2(R)}. \quad (8)$$

Compared to that, the two-loop fixed point value of the coupling constant reads [7]

$$\frac{\alpha^*}{4\pi} = -\frac{\beta_0}{\beta_1}. \quad (9)$$

For a fixed number of colours the critical number of flavours for which the order of  $\alpha_*$  and  $\alpha_c$  changes is defined by imposing  $\alpha_* \stackrel{!}{=} \alpha_c$ , and is given by

$$N_f^{II}[R] = \frac{d(G)C_2(G)}{d(R)C_2(R)} \frac{17C_2(G) + 66C_2(R)}{10C_2(G) + 30C_2(R)}. \quad (10)$$

In order to evaluate the above expressions and throughout the article we use the Dynkin indices (the Dynkin labels of the highest weight of an irreducible representation) to uniquely characterise the representations and determine the relevant coefficients. We summarise, for the reader's convenience, the relevant formulae in the appendix A.

<sup>1</sup> Notice the different normalisation as compared to, for example [7]. Conveniently, below, the quadratic Casimir operator will only assume integer values. Nevertheless, many of the subsequent expressions turn out to be identical due to the ratios that are taken.

<sup>2</sup> We investigate here whether chiral symmetry breaking occurs in a *walking* theory. Thus, corrections to the ladder approximation like vertex corrections, which take into account the change of the coupling constant, are small. Alternative methods for calculating the conformal window have been provided, e.g. in [5, 17, 37, 40].

## B. Classification of asymptotically free theories

As we increase the dimension of the representation of the matter fields the screening effect of the matter compensates more and more the anti-screening effect of the gauge bosons. Eventually one loses asymptotic freedom. Here we will render quantitative this fact by constructing all of the theories which are asymptotically free with at least two Dirac flavours.

For two flavours the fundamental, adjoint, as well as two-index symmetric and antisymmetric representations remain asymptotically free for any number of colours. We will show below that there are no exceptions from this rule for more than nine colours. Apart from theories based on these representations the only remaining variants are the three-index antisymmetric representation which is asymptotically free with two flavours for six to nine colours<sup>3</sup> and the four-index antisymmetric with two flavours for eight colours. For  $N \leq 9$  all such theories are listed in Table I.

That at a large number of colours only the fundamental, adjoint, two-index symmetric, and two-index antisymmetric representations survive is due to the fact that they are the only ones whose dimension grows quadratically or more slowly with the number of colours. It can thus be compensated by the quadratic growth of the dimension of the adjoint representation in the expressions for  $N_f^I$ . For any given representation the quadratic Casimir operators grow quadratically with (large)  $N$  and their ratio for different representations goes to a constant.

### 1. Only four remaining representations for $N > 9$

In this subsection we prove that

*There exist no asymptotically free theories with at least two flavours and ten or more colours which are not contained in the fundamental, adjoint, or the two-index representations.*

The utilised methods are general and can be applied to different numbers of Dirac flavours. We will consider also the case of one flavour. We start by analysing the ordering of  $N_f^I(R)$  at fixed  $N$  for different representations, R.

*Ordering  $N_f^I(R)$  at fixed  $N$*

As all coefficients in Eqs. (A1) and (A2) are positive, we have, at a fixed number of colours,  $N$  (that is with a

R	$\bar{R}$	$N_f^I$	$N_f^{II}$	$N_f^{III}$	$\pi S$	$\lambda_*$
(1)	$\equiv$	11	$7\frac{73}{85}$	$5\frac{27}{49}$	1	3.229
(2)	$\equiv$	$2\frac{3}{4}$	$2\frac{3}{40}$	$1\frac{1}{16}$	$\frac{1}{2}$	1,124.
(10)	(01)	$16\frac{1}{2}$	$11\frac{32}{35}$	$8\frac{1}{19}$	$2\frac{1}{2}$	10.33
(20)	(02)	$3\frac{3}{10}$	$2\frac{163}{325}$	$1\frac{28}{125}$	1	11.29
(11)	$\equiv$	$2\frac{3}{4}$	$2\frac{3}{40}$	$1\frac{1}{16}$	$1\frac{1}{3}$	1,124.
(100)	(001)	22	$15\frac{361}{385}$	$10\frac{126}{205}$	$4\frac{2}{3}$	20.70
(200)	(002)	$3\frac{2}{3}$	$2\frac{82}{105}$	$1\frac{71}{201}$	$1\frac{2}{3}$	5.865
(010)	$\equiv$	11	$8\frac{12}{115}$	$4\frac{52}{55}$	4	127,359.
(101)	$\equiv$	$2\frac{3}{4}$	$2\frac{3}{40}$	$1\frac{1}{16}$	$2\frac{1}{2}$	1,124.
(1000)	(0001)	$27\frac{1}{2}$	$19\frac{58}{61}$	$13\frac{32}{161}$	$7\frac{1}{2}$	35.90
(2000)	(0002)	$3\frac{13}{14}$	$2\frac{747}{763}$	$1\frac{662}{1463}$	$2\frac{1}{2}$	4.284
(0100)	(0010)	$9\frac{1}{6}$	$6\frac{191}{237}$	$3\frac{514}{537}$	5	29.00
(1001)	$\equiv$	$2\frac{3}{4}$	$2\frac{3}{40}$	$1\frac{1}{16}$	4	1,124.
(10000)	(00001)	33	$23\frac{283}{295}$	$15\frac{123}{155}$	11	57.57
(20000)	(00002)	$4\frac{1}{8}$	$3\frac{33}{260}$	$1\frac{53}{100}$	$3\frac{1}{2}$	2.519
(01000)	(00010)	$8\frac{1}{4}$	$6\frac{3}{20}$	$3\frac{21}{44}$	$7\frac{1}{2}$	5,146.
(00100)	$\equiv$	$5\frac{1}{2}$	$4\frac{18}{145}$	$2\frac{14}{61}$	$6\frac{2}{3}$	2,198.
(10001)	$\equiv$	$2\frac{3}{4}$	$2\frac{3}{40}$	$1\frac{1}{16}$	$5\frac{5}{6}$	1,124.
(100000)	(000001)	$38\frac{1}{2}$	$27\frac{584}{605}$	$18\frac{125}{317}$	$30\frac{1}{3}$	87.67
(200000)	(000002)	$\frac{77}{18}$	$3\frac{2294}{9495}$	$1\frac{2168}{3663}$	$4\frac{2}{3}$	3.061
(010000)	(000010)	$7\frac{7}{10}$	$5\frac{3186}{4225}$	$3\frac{356}{1825}$	7	4.050
(001000)	(000100)	$3\frac{17}{20}$	$2\frac{8707}{9650}$	$1\frac{1941}{3890}$	$5\frac{5}{6}$	4.421
(100001)	$\equiv$	$2\frac{3}{4}$	$2\frac{3}{40}$	$1\frac{1}{16}$	8	1,124.
(1000000)	(0000001)	44	$31\frac{1537}{1585}$	$20\frac{828}{829}$	40	128.6
(2000000)	(0000002)	$4\frac{2}{5}$	$3\frac{1141}{3425}$	$1\frac{851}{1325}$	6	2.752
(0100000)	(0000010)	$7\frac{1}{3}$	$5\frac{829}{1695}$	$3\frac{7}{723}$	$9\frac{1}{3}$	5.329
(0010000)	(0000100)	$2\frac{14}{15}$	$2\frac{8738}{39975}$	$1\frac{1733}{15675}$	$9\frac{1}{3}$	52.90
(1000001)	$\equiv$	$2\frac{3}{4}$	$2\frac{3}{40}$	$1\frac{1}{16}$	$10\frac{1}{2}$	1,124.
(10000000)	(00000001)	$49\frac{1}{2}$	$35\frac{326}{335}$	$23\frac{106}{157}$	51	183.4
(20000000)	(00000002)	$4\frac{1}{2}$	$3\frac{516}{1265}$	$1\frac{1678}{2453}$	$7\frac{1}{2}$	2.526
(01000000)	(00000010)	$7\frac{1}{14}$	$5\frac{1016}{3395}$	$2\frac{1261}{1435}$	12	6.669
(10000001)	(10000001)	$2\frac{3}{4}$	$2\frac{3}{40}$	$1\frac{1}{16}$	$13\frac{1}{3}$	1,124.

TABLE I: Complete list of asymptotically free  $SU(N)$  theories with at least one family of fermions and up to nine colours.  $\lambda_*$  [Eq. (18)] and  $\pi S$  [Eq. (17)] are calculated for  $N_f < N_f^{II}$  and even.

fixed number of Dynkin indices),

$$a_j \geq b_j \quad \forall j \in \{1; \dots; N\} \Rightarrow C_2(\{a_j\}) \geq C_2(\{b_j\})$$

$$\text{and } d(\{a_j\}) \geq d(\{b_j\}). \quad (11)$$

<sup>3</sup> For a number of colours less than six the three-index antisymmetric always coincides with another representation with a smaller number of indices.

	$d(R)$	$C_2(R)$	$N_f^I$	$N_f^{II}$	$N_f^{III}$
<b>F</b>	$N$	$N^2 - 1$	$\frac{11}{2}N$	$\frac{2}{5}N \frac{50N^2-33}{5N^2-3}$	$\frac{34N^3}{13N^2-3}$
<b>G</b>	$N^2 - 1$	$2N^2$	$2\frac{3}{4}$	$2\frac{3}{40}$	$1\frac{1}{16}$
$S_n$	$\frac{(N+n-1)!}{n!(N-1)!}$	$n(N-1)(N+n)$	$\frac{11N(n-1)!(N+1)!}{2(n+N)!}$	$\frac{2N(33n^2(N-1)+33nN(N-1)+17N^2)n!(N+1)!}{5n[3n^2(N-1)+3nN(N-1)+2N^2](n+N)!}$	$\frac{34N^3(n-1)!(N+1)!(n+N)!^{-1}}{3n^2(N-1)+3nN(N-1)+10N^2}$
<b>S<sub>2</sub></b>	$\frac{N(N+1)}{2}$	$2(N-1)(N+2)$	$\frac{11}{2} \frac{N}{N+2}$	$\frac{N}{N+2} \frac{83N^2+66N-132}{20N^2+15N-30}$	$\frac{17N^3}{(N+2)(8N^2+3N-6)}$
<b>S<sub>3</sub></b>	$\frac{N(N+1)(N+2)}{6}$	$3(N-1)(N+3)$	$\frac{11N}{(N+2)(N+3)}$	$\frac{4N(-297+2N(99+58N))(N+1)!}{5(-27+N(18+11N))(N+3)!}$	$\frac{68N^3(N+1)!}{(-27+N(18+19N))(N+3)!}$
$A_n$	$\frac{N!}{n!(N-n)!}$	$n(N-n)(N+1)$	$\frac{11}{2}N \binom{N-2}{n-1}^{-1}$	$\binom{N-2}{n-1}^{-1} \frac{(34+66n)N^3-66n(n-1)N^2-66n^2N}{(10+15n)N^2-15n(n-1)N-15n^2}$	$\binom{N-2}{n-1}^{-1} \frac{34N^3}{10N^2-3n^2(N+1)+3nN(N+1)}$
<b>A<sub>2</sub></b>	$\frac{N(N-1)}{2}$	$2(N-2)(N+1)$	$\frac{11}{2} \frac{N}{N-2}$	$\frac{N}{N-2} \frac{83N^2-66N-132}{20N^2-15N-30}$	$\frac{17N^3}{(N-2)(8N^2-3N-6)}$
<b>A<sub>3</sub></b>	$\frac{N(N-1)(N-2)}{6}$	$3(N-3)(N+1)$	$\frac{11N}{(N-2)(N-3)}$	$\frac{4N}{(N-2)(N-3)} \frac{116N^2-198N-297}{55N^2-90N-105}$	$n = 3$ in $A_n$ above
<b>A<sub>4</sub></b>	$\frac{N(N-1)(N-2)(N-3)}{24}$	$4(N-4)(N+1)$	$\frac{33N}{(N-2)(N-3)(N-4)}$	$\frac{6N}{(N-2)(N-3)(N-4)} \frac{149N^2-396N-528}{35N^2-90N-120}$	$n = 4$ in $A_n$ above
<b>R<sub>1</sub></b>	$\frac{(N-1)N^2(N+1)}{12}$	$4(N^2-4)$	$\frac{33}{2(N^2-4)}$	$\frac{3(149N^2-528)}{5(7N^4-52N^2+96)}$	$\frac{51N^2}{11N^4-68N^2+96}$
<b>R<sub>2</sub></b>	$\frac{(N-1)N(N+1)}{3}$	$3(N^2-3)$	$\frac{11N}{2(N^2-3)}$	$\frac{2N(-297+116N^2)}{5(81-60N^2+11N^4)}$	$\frac{34N^3}{81-84N^2+19N^4}$
<b>R<sub>3</sub></b>	$\frac{(N+1)N(N-2)}{2}$	$(3N+1)(N-1)$	$\frac{11N}{(3N+1)(N-2)}$	$\frac{4N(-33-66N+116N^2)}{5(6+27N-N^2-73N^3+33N^4)}$	$\frac{68N^3}{6+27N-17N^2-113N^3+57N^4}$
<b>R<sub>4</sub></b>	$\frac{(N+1)N^2(N-3)}{4}$	$4N(N-1)$	$\frac{11}{2N(N-3)}$	$\frac{149N-132}{5N(N-3)(7N-6)}$	$\frac{17}{18-39N+11N^2}$

TABLE II: Characteristic quantities sorted after representations: F=fundamental, G=adjoint,  $S_n=n$ -index symmetric,  $A_n=n$ -index antisymmetric,  $R_1=(020\dots 0)$ ,  $R_2=(110\dots 0)$ ,  $R_3=(10\dots 010)$ ,  $R_4=(010\dots 010)$ . Representations marked in boldface lead to theories that stay asymptotically free with at least two flavours for any number of colours.

Using Eq. (6) we find

$$a_j \geq b_j \quad \forall j \in \{1; \dots; N\} \Rightarrow N_f^I(\{a_j\}) \leq N_f^I(\{b_j\}). \quad (12)$$

By increasing the value of any of the Dynkin indices the critical number of flavours decreases. The next step is to determine a complete set of theories with  $N_f^I(R)$  just below two. They are to serve as universal bounds on  $N_f^I(R)$  for all theories.

$n \geq 3$ -index antisymmetric representations

Let us start from the  $n \geq 3$ -index antisymmetric representations,  $A_n(N)$ . We notice from the corresponding expression for  $N_f^I[A_n(N)]$  in Table II that for  $N \geq 2n$ ,<sup>4</sup>

$$N_f^I[A_n(N)] > N_f^I[A_n(N+1)] \quad (13)$$

<sup>4</sup> We are in the right-hand flank of Pascal's triangle. For  $N < 2n$  the  $n$ -index antisymmetric representation is covered by an  $(n' < n)$ -index antisymmetric representation due to the symmetry of the result under conjugation, that is inversion of the order of the Dynkin indices. For example (0010), which is  $A_3$ , is, in this sense, equivalent to (0010) = (0100) which is  $A_2$ .

and

$$N_f^I[A_{n-1}(N)] > N_f^I[A_n(N)]. \quad (14)$$

We deduce that  $N_f^I$  shrinks if the number of colours,  $N$ , or the number of indices is increased as long as  $N \geq 2n$ .

Therefore, given a reference value  $N_f^I[A_{n_0}(N_0)]$ ,

$$(n > n_0 \text{ or } N > N_0) \Rightarrow N_f^I[A_n(N)] < N_f^I[A_{n_0}(N_0)]. \quad (15)$$

Picking as reference value  $N_f^I[A_3(10)] = 1\frac{27}{28}$ , we know that

$$N_f^I[A_n(N)] \leq 1\frac{27}{28} \quad \forall n \geq 3, N \geq 10. \quad (16)$$

For a given number of colours,  $N$ , starting from all antisymmetric representations from  $A_3(N)$  to  $A_{N-3}(N)$ , we can reach other representations by *increasing* the values of the Dynkin indices. The Dynkin representation of an  $n$ -index antisymmetric representation reads  $(0\dots 010\dots 0)$ . Therefore, the only representations that cannot be reached in this way are  $(a_1a_20\dots 0a_{N-2}a_{N-1}) \quad \forall a_1, a_2, a_{N-2}, a_{N-1}$ . In combination with the findings of the previous subsection this leads to the conclusion that only these latter representations may, but need not, have  $N_f^I[A_n(N)] > 1\frac{27}{28}$ .

### $S_3$ and $(020\dots 000)$

As can be seen from Table I we cannot obtain an equally low boundary from the fundamental,  $(100\dots 000)$ , and the two-index antisymmetric representation,  $(010\dots 000)$ . The same holds for the two-index symmetric representation,  $(200\dots 000)$ . Hence, we continue with the three-index symmetric representation,  $S_3(N)=(300\dots 000)$ , and the representation  $R_1(N):=(020\dots 000)$ .  $N_f^I[S_3(N)]$  and  $N_f^I[R_1(N)]$  are monotonically decreasing functions of the number of colours. This can be seen from the explicit expressions in Table II. As reference values, we can pick  $N_f^I[S_3(3)] = 1\frac{1}{10}$  and  $N_f^I[R_1(4)] = 1\frac{3}{8}$ , which are both smaller than the previous reference value  $N_f^I[A_3(10)] = 1\frac{27}{28}$  and are situated at a smaller number of colours as well. The representations of the set  $\{(a_1 a_2 0\dots 0 a_{N-2} a_{N-1}) \mid \forall a_1, a_2, a_{N-2}, a_{N-1} \in \mathbb{N}\}$ , which cannot be obtained by increasing the Dynkin indices either of  $S_3(N)$  or of  $R_1(N)$  or of their conjugates are given by  $(a_1 a_2 0\dots 0 a_{N-2} a_{N-1}) \mid \forall (a_1, a_{N-1} \in \{0; 1; 2\} \text{ and } a_2, a_{N-2} \in \{0; 1\})$ . This amounts to 36 combinations before making use of symmetry properties under conjugation.

$(110\dots 000)$ ,  $(100\dots 010)$ , and  $(010\dots 010)$

In the next step, we exploit information on the representations  $R_2(N)=(110\dots 000)$ ,  $R_3(N)=(100\dots 010)$ , and  $R_4(N)=(010\dots 010)$  in order to set limits for most of the remaining representations. To this end, we repeat the same steps as before. First we check that  $N_f^I$  for these representations is a decreasing function of the number of colours. This can be seen directly from the corresponding expressions in Table II. Subsequently, we pick reference values for the three cases:  $N_f^I[R_2(4)] = 1\frac{9}{13}$ ,  $N_f^I[R_3(4)] = 1\frac{9}{13}$ , and  $N_f^I[R_4(4)] = 1\frac{3}{8}$ , respectively. They are all smaller than  $N_f^I[A_3(10)] = 1\frac{27}{28}$  and lie at a smaller number of colours. The limits set by these reference values cannot be exceeded for a larger number of colours in these representations. Finally, we eliminate all representations from the remaining 36 that can be generated by adding to the Dynkin indices of  $R_2$ ,  $R_3$ , or  $R_4$ . This leaves but  $\overline{F}$ ,  $\overline{G}$ ,  $A_2$ , and  $S_2$  as well as their conjugates  $\overline{F}$ ,  $\overline{A_2}$ , and  $\overline{S_2}$  ( $G$  is a real representation), that is seven representations of which four are independent.

Thereby we have shown that beyond ten colours no other representations lead to asymptotically free theories with two flavours but the fundamental, the adjoint, as well as the two-index symmetric and antisymmetric representations. Below ten colours the only exceptions are given by the three-index antisymmetric representation at six to nine colours and the four-index antisymmetric at eight colours. This we have checked by explicit calculation and again by making use of the fact that  $N_f^I$  increases if any Dynkin index is increased.

As  $N_f^I > N_f^{II} > N_f^{III}$  the above findings are handed

down to the lower bound for the conformal window and the existence of a Banks-Zaks fixed point.

### 2. One flavour

When the requirement is weakened to asymptotic freedom with one flavour, all the above steps can be repeated after choosing new reference values:  $N_f^I[A_3(16)] = \frac{88}{91}$ ,  $N_f^I[S_3(5)] = \frac{55}{56}$ ,  $N_f^I[R_1(5)] = \frac{11}{14}$ ,  $N_f^I[R_2(7)] = \frac{77}{92}$ ,  $N_f^I[R_3(6)] = \frac{33}{38}$ ,  $N_f^I[R_4(5)] = \frac{11}{20}$ . This means that for the one flavour limit, no exceptions are present from 16 colours onward.

### C. Conformal window

Evaluating the expression (10) for the lower bound of the conformal window for all theories, which are asymptotically free with at least two flavours, leads to the values listed in the fourth column of Table I. For the fundamental, adjoint and two-index symmetric as well as antisymmetric representations there exist theories with at least two flavours, which have not yet entered the conformal phase. This remains true beyond nine colours. From the exceptions below ten colours, the three-index antisymmetric representation at nine colours and the four-index antisymmetric representations are already conformal with less than two flavours.

The minimal number of flavours necessary for a Banks-Zaks fixed point to appear as calculated from Eq. (7) is listed in the fifth column of Table I. For values beyond nine colours, the explicit expressions in Table II can be evaluated. The present analysis exhausts the phase diagram for gauge theories with Dirac fermions in arbitrary representations as function of the number of colours and flavours.

## III. WALKING TECHNICOLOUR

Here we will use our findings concerning asymptotically free theories with at least one doublet of fermions to identify all physically acceptable technicolour models. In technicolour models the dynamical breaking of the electroweak symmetry from  $SU_L(2) \times U_Y(1)$  to  $U_{em}(1)$  is not generated by an elementary Higgs particle like in the standard model, but by chiral symmetry breaking in an additional strongly interacting sector. It is made up of techniquarks transforming under the electroweak and an additional technicolour gauge group. The scale at which the chiral symmetry of the technicolour sector breaks is chosen to be the electroweak scale. Three of the emergent Goldstone bosons are absorbed as longitudinal degrees of freedom of the electroweak gauge bosons, which thus become massive. The fermion masses are generated by embedding the electroweak and technicolour gauge



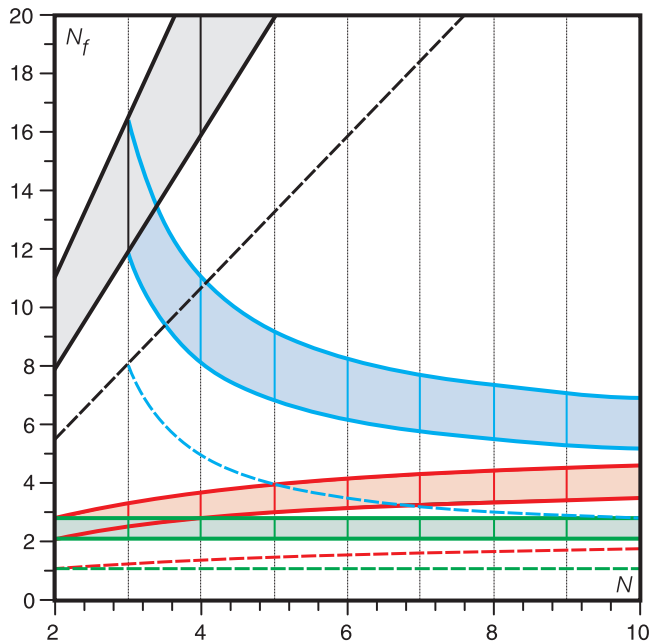


FIG. 1: Phase diagram for theories with fermions in the (from top to bottom in the plot; colour online): i) fundamental representation (grey), ii) two-index antisymmetric (blue), iii) two-index symmetric (red), iv) adjoint representation (green) as a function of the number of flavours and the number of colours. The shaded areas depict the corresponding conformal windows. The upper solid curve represents  $N_f^I[R(N)]$  (loss of asymptotic freedom), the lower  $N_f^{II}[R(N)]$  (loss of chiral symmetry breaking). The dashed curves show  $N_f^{III}[R(N)]$  (existence of a Banks–Zaks fixed point). Note how consistently the various representations merge into each other when, for a specific value of  $N$ , they are actually the same representation.

groups in a larger extended technicolour gauge group. The gauge bosons of the extended technicolour model couple the fermions of the standard model to the techniquarks and their condensate, which renders the standard model fermions massive.

Like all other mechanisms for electroweak symmetry breaking, technicolour has to face constraints derived from experimental data. In the case of technicolour the two main aspects are additional contributions to the vacuum polarisation of the electroweak gauge bosons (oblique parameters) and flavour changing neutral currents as well as lepton number violation due to the extended technicolour dynamics. These issues have been discussed in great detail in the literature (see, for example [20, 21]). Experimental data (see, for example [22, 23]) tells us that the above mentioned contributions must be small. Here, let us only recall that flavour changing neutral currents and lepton number violation are suppressed in walking technicolour theories, that is technicolour theories with nearly conformal dynamics. Through non-perturbative effects, quasi-conformality also helps reducing the techniquarks’ contribution to the oblique parameters [11, 12, 13, 14, 24]. (In the absence of quasi-

conformal dynamics the  $S$  parameter can be larger than its perturbative value.) On top of that, potential additional Goldstone bosons, beyond the three which are absorbed as the longitudinal degrees of freedom of the electroweak gauge bosons, become very heavy, thereby alleviating bounds set by them not having been detected to date. Therefore, candidates for realistic technicolour theories should feature quasi-conformal dynamics and should contribute little to the oblique parameters already at the perturbative level. In what follows, we will quantify these criteria.

Already taking into account the experimental limits on the  $S$ -parameter [26] severely constrains the set of candidates. Perturbatively, it is given by

$$S = \frac{1}{6\pi} \frac{N_f}{2} d(R). \quad (17)$$

The values for  $S$  are given in Table I. Drawing the line at  $S < \pi^{-1}$ —somewhat arbitrarily but in accordance with the experimental limits [22, 23]—leaves us with three candidates which, characterised by their Dynkin indices are: (1) with six flavours, (2) with two flavours, and (20) with two flavours. Doubling the value of the cut on the  $S$  parameter ( $S < 2\pi^{-1}$ ) would admit two more: (11) with two flavours and (200) with two flavours.

The estimate for the lower bound (critical number of flavours) of the conformal window is based on the point where the critical coupling and the fixed point value coincide. This critical number of flavours is, in general, not an even integer. A quasi-conformal physical realisation of a technicolour theory is, however, constructed from complete families of techniquarks.<sup>5</sup>

From the difference of the two scales, the amount of walking, that is the ratio of the scale can be estimated [8, 11],

$$\lambda_* \approx \exp(\pi/\sqrt{\alpha_*/\alpha_c - 1}). \quad (18)$$

$\lambda_*$  is the ratio of the scale from which onwards the coupling constant stays approximately constant divided by the scale for which it starts running again. For this walking mechanism to be effective it must typically cover several decades. Setting the cut at  $\lambda_* > 10^3$  leaves (2) with two flavours (see Table I). [If the weaker bound on the  $S$  parameter is chosen also (11) with two flavours survives.] Weakening the requirement on the range of the walking to  $\lambda_* > 10^2$  leads to no supplementary candidates.

#### 1. Two flavours, $SU(2)$ , adjoint representation: (2)

The technicolour theory with two techniquarks in the two-index symmetric/adjoint representation of  $SU(2)$ ,

<sup>5</sup> Generalisations with an odd number of Dirac or even Weyl flavours are conceivable. A corresponding example is given in Sect. IIIC.

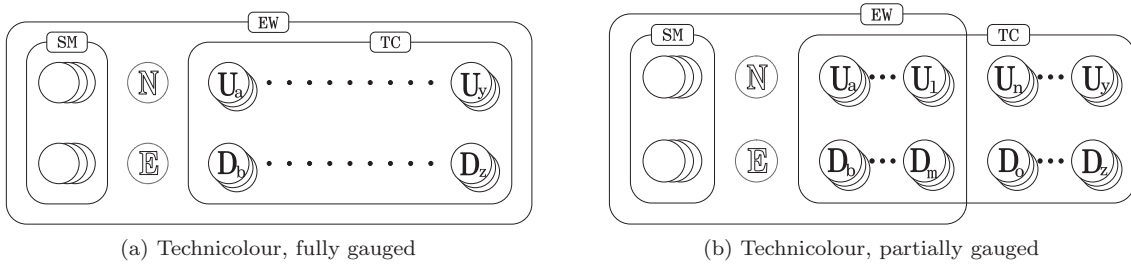


FIG. 2: **Technicolour models.** The boxes depict under which gauge groups the different particles transform. The box headlined “SM” represents all standard model particles (excluding an elementary Higgs). N and E stand for a fourth family of leptons, which may or may not have to be included in order to evade a topological Witten anomaly. They have to be included, if the number of techniquark families transforming under the electroweak symmetry times the number of colour degrees of freedom is an odd number. *Left panel:* In fully gauged technicolour models, all techniquarks transform under the electroweak symmetry. *Right panel:* In partially gauged technicolour a part of the techniquarks are electroweak singlets. It is conceivable that only one (maximal splitting) or several (sub-maximal splitting) families of techniquarks carry electroweak charges. The latter set-up may be an alternative cure from the Witten anomaly not involving additional leptons.

that is (2), has been studied in [9, 10, 27] and found to be in good agreement with experimental constraints. It has to contain an additional family of leptons in order to avoid the topological Witten anomaly<sup>6</sup>[28]. This model has a rich phenomenology owing to the  $SU(2N_f=4)$  flavour symmetry in the unbroken phase, which is enhanced because the matter transforms under the adjoint representation, which is real. The breaking to  $SO(4)$  leads to nine Goldstone bosons, three of which become the longitudinal degrees of freedom of the electroweak gauge bosons. The remaining six are of technibaryonic nature and may be very massive due to the intense walking of the theory. They can contribute to dark matter as might one of the additional leptons. This always depends on the hypercharge assignment for the particles beyond the standard model, which here is not fixed totally by requiring the freedom from gauge anomalies. Since, in the present theory, the fermions transform under the adjoint representation, their technicolour can be neutralised directly by techniguons, which leads to potentially phenomenologically interesting states. They are expected to have a mass of the order of the confining scale of the theory. For more details on this particular candidate see [27].

## 2. Two flavours, $SU(3)$ , adjoint representation: (11)

The theory with two techniflavours in the adjoint representation of  $SU(3)$ , (11), features a similarly enhanced flavour symmetry,  $SU(2N_f=4)$ , whose breaking to  $SO(4)$  gives rise to a total of nine Goldstone bosons with the

same consequences as in the two-colour case. The theory does not have to contain additional leptons whence the hypercharge assignment is fixed such that the techniquarks carry half-integer electrical charges with opposite signs. All objects of two and more techniquarks can be technicolour neutral and the technicolour of even a single techniquark can be neutralised by techniguons. This is in direct analogy to the two-technicolour case.

## A. Partially gauged technicolour

A small modification of the traditional technicolour approach, which neither involves additional particle species nor more complicated gauge groups, allows constructing several other viable candidates. It consists in letting only one doublet of techniquarks transform non-trivially under the electroweak symmetries with the rest being electroweak singlets, as already suggested in [9] and later used in [29]. Still, all techniquarks transform under the technicolour gauge group. Thereby, perturbatively, only one techniquark doublet contributes to the oblique parameter which is thus kept to a minimum for theories which need more than one family of techniquarks to be quasi-conformal. It is the condensation of that first electroweakly charged family that breaks the electroweak symmetry. Additionally, the number of ungauged techniquarks, in general, need not be even. In certain cases, this allows to come closer to conformality (see Sect III B 1). There exist several more partially gauged candidates with  $S < \pi^{-1}$  (see Table I): (1) with seven flavours [instead of six fully gauged flavours], (10) with eleven flavours, (100) with 15 flavours, (010) with eight flavours, (1000) with 19 flavours, and (10000) with 23 flavours. Of these last-mentioned theories with techniquarks in the fundamental representation, (10...0), only those with four or more colours walk over more than two decades. Three decades could be reached from seven technicolours and 27 techniflavours onwards, which leads to a perturba-

<sup>6</sup> It does not allow for an odd number of families of fermions to transform under an  $SU(2)$  gauge group [here the  $SU_L(2)$  of the electroweak gauge group]. The one family of (2), however, corresponds to three [ $\dim(2)=3$ ] technicolour copies added to the previously even number of SM fermion families.

tive contribution to the  $S$  parameter of  $\pi S \approx 1.2$  from the two electroweakly gauged technifermions. Therefore, the parametrically admissible models in the fundamental representation are rather non-minimalist.

The theory with eight techniquarks in the two-index antisymmetric representation of  $SU(4)$ , (010), on the other hand, according to Eq. (18) walks over more than five decades.

The techniquarks which are uncharged under the electroweak gauge group are natural building blocks for components of dark matter.

*Eight flavours,  $SU(4)$ , two-index antisymmetric representation: (010)*

Among the partially gauged cases the prime candidate is the theory with eight techniflavours in the two-index antisymmetric representation of  $SU(4)$ . The techniquarks of one of the four families carry electroweak charges while the others are electroweak singlets. Gauge anomalies are avoided if the two electrically charged techniquarks possess half-integer charges. The technihadron spectrum can contain technibaryons made of only two techniquarks because in the two-index antisymmetric representation of  $SU(4)$  a singlet can already be formed in that case  $[(010) \otimes (010) \rightarrow (000) \oplus (101) \oplus (020)]$ . Otherwise technisinglets can also be formed from four techniquarks. All technihadrons formed from techniquarks without electrical charges can contribute to dark matter. Due to the special charge assignment of the electrically charged particles (opposite half-integer charges) certain combinations of those can also be contained in electrically uncharged technibaryons. For instance we can construct the following completely neutral technibaryon:

$$\epsilon_{t_1 t_2 t_3 t_4} Q_L^{t_1 t_2, f} Q_L^{t_3 t_4, f'} \epsilon_{ff'} , \quad (19)$$

where  $\epsilon_{ff'}$  saturates the  $SU(2)_L$  indices of the two gauged techniquarks and the first antisymmetric tensor  $\epsilon$  is summed over the technicolour indices. We have suppressed the spin indices. If there is no technibaryon asymmetry the cross section for annihilation would be too big and the present relic density would be negligible. In the presence of a technibaryon asymmetry, however, this particle would be another candidate for dark matter (different from those formed from electroweakly neutral techniquarks), hardly detectable in any earth based experiment [27].

As (010) is a real representation the model's flavour symmetry is enhanced to  $SU(2N_f=16)^7$ , which, when it

breaks to  $SO(16)$ , induces 135 Goldstone bosons<sup>8</sup>.

It is worth recalling that the centre group symmetry left invariant by the fermionic matter is a  $Z_2$  symmetry. Hence there is a well defined order parameter for confinement [30] which can play a role in the early Universe.

## B. Beyond the prime candidate

In view of the uncertainty of the experimental limits for the oblique parameters (see, for example [31]) and the approximations made here to determine the degree of walking, not all the other theories, which above have not been identified as prime candidates for a realistic walking technicolour theory, should be considered as being ruled out. While the theories which, in this analysis, stood out as most favoured, would, with the highest likelihood, survive an amelioration of the experimental limits and/or a refinement of the theoretical assessment, others might emerge as being viable candidates as well. For these reasons, we address some of those in the rest of this section. At the end, we will have addressed all settings listed in Table III.

### 1. Fundamental

Compared to the prime examples with matter in higher-dimensional representations, models with fundamental matter must have a comparatively large number of colours and flavours in order to feature a sufficient amount of walking. In the partially gauged approach their  $S$  parameter reaches  $\pi^{-1}$  for six colours. Phenomenologically, the models with fundamental techniquarks can be divided into two groups with an even, respectively odd number of colours (=dimension). Technicolour singlets are always formed by  $N$  techniquarks.

For an even number of colours the model with one electroweakly gauged techniquark doublet is free of the Witten anomaly. No additional lepton family has to be included. Opposite half-integer electrical charges for the technifermions avoid gauge anomalies. Technicolour singlets constructed from the techniquarks which are gauged under the electroweak interactions carry integer electrical charges. This also includes uncharged technibaryons which can be components of dark matter.

For an odd number of colours a single doublet of techniquarks gauged under the electroweak interactions leads to a Witten anomaly. It can be circumvented by including one additional lepton family. This allows for a more general hypercharge assignment [9] than in the case without leptons.

<sup>7</sup> It is slightly explicitly broken by the electroweak interactions. This is, of course, always the case between the techniups and technidowns, but here, additionally, there is a difference, on the electroweak level, between gauged and ungauged techniquark families.

<sup>8</sup> Obviously the Goldstone must receive a sufficiently large mass. This is usually achieved in extended technicolour. Still, they could be copious at LHC.



R	$N_f$	flavour symmetry	WA	P-G	$\lg \lambda_*$	$\pi S^a$
(1)	7	$SU(14) \rightarrow Sp(14)$	no	yes	1.3	$0.3^b$
<b>(2)</b>	2	$SU(4) \rightarrow SO(4)$	yes	–	3.1	0.5
(10)	11	$SU_L(11) \times SU_R(11)$	yes	yes <sup>c</sup>	1.8	0.5
(20)	2	$SU_L(2) \times SU_R(2)$	no	–	1.1	1.0
(11)	2	$SU(4) \rightarrow SO(4)$	no	–	3.1	1.3
(100)	15	$SU_L(15) \times SU_R(15)$	no	yes	2.2	0.7
(200)	2	$SU_L(2) \times SU_R(2)$	no	–	0.8	1.7
<b>(010)</b>	8	$SU(16) \rightarrow SO(16)$	no	yes	5.1	1.0
(101)	2	$SU(4) \rightarrow SO(4)$	yes	–	3.1	2.5
(1000)	19	$SU_L(19) \times SU_R(19)$	yes	yes <sup>c</sup>	2.4	0.8
(0100)	6	$SU_L(6) \times SU_R(6)$	no	yes	1.5	1.7
(10000)	23	$SU_L(23) \times SU_R(23)$	no	yes	2.7	1.0

<sup>a</sup>The contribution to  $S$  from additional leptons is not included. It is of the order  $\pi S \approx 1/6$ . The contribution to the  $T$  parameter from non-mass-degenerate leptons helps adjusting to the correlation of the  $S$  and  $T$  parameters indicated by experimental data [9, 10].

<sup>b</sup>This value results for a partial weak gauging involving only one doublet of the theory. If all possible, that is three doublets are gauged one has  $\pi S \approx 1$ .

<sup>c</sup>In these cases, the global Witten anomaly can be circumvented either by adding one family of leptons or by having two families of techniquarks charged under the electroweak interactions. The value for the  $S$  parameter is for the former case. In the latter, it is twice as large.

TABLE III: List of possible technicolour theories. The first two columns give the representation and the number of flavours, respectively. In the next column follows the symmetry breaking pattern. [ $SU_L(N_f) \times SU_R(N_f)$  always breaks to  $SU(N_f)$ .] Thereafter, it is marked whether a global Witten anomaly has to be taken care of by adding a family of leptons or, where this is possible, by a less minimal partial weak gauging of the theory. Finally, the amount of walking and the perturbative  $S$  parameter are listed. The prime candidates, that is the theories with the largest amount of walking at an acceptable size for the oblique corrections are marked in boldface.

An alternative to an additional lepton family for circumventing a Witten anomaly is to gauge two instead of one techniquark doublet, at the cost of doubling the  $S$  parameter. For this non-minimal weak gauging the hypercharge assignment which corresponds to opposite half-integer electric charges for the techniquarks avoids gauge anomalies.

## 2. Two-index symmetric

Two more theories with apparently insufficient walking but which make at least the weaker bounds on the  $S$  parameter are the two flavour theories with techniquarks in the two-index symmetric representation of  $SU(3)$ , (20), and  $SU(4)$ , (200). None of the two needs additional leptons. Thus the hypercharge assignment is completely

fixed. Three respectively four techniquarks are needed to form a singlet technibaryon.

## 3. Two flavours, $SU(4)$ , adjoint

The four technicolour model with two techniflavours in the adjoint representation has a relatively large perturbative  $S$  parameter but walks over more than three decades which helps reducing it through non-perturbative corrections. It also requires an additional lepton family in order to cure the Witten anomaly, which allows for a more general hypercharge assignment. The enhanced symmetry,  $SU(2N_f=4)$ , breaks to  $SO(4)$  and leaves behind nine Goldstone bosons. They must be rendered massive which also necessitates a good amount of walking. As for all other adjoint models, any number of techniquarks can form a singlet and technigluons can neutralise technicolour as well.

## 4. Six flavours, $SU(5)$ , two-index antisymmetric

Finally, let us mention the five technicolour model with six techniflavours in the two-index antisymmetric representation. For an acceptable  $S$  parameter, only part of the techniflavours can be gauged under the electroweak interactions. No additional leptons are required. This variant does not exhibit a remarkable amount of walking.

## C. Split technicolour: A review

Since one aim of this work is to provide a catalogue of various possible walking type, non-supersymmetric gauge theories, which can be used to dynamically break the electroweak symmetry we summarize here also another possibility already appearing in [9]. There we also suggested a way of keeping the technifermions in the fundamental representation while still reducing the number of techniflavours needed to be near the conformal window. Like for the partially gauged case described above this can be achieved by adding matter uncharged under the weak interactions. The difference to section III A is that this part of matter transforms under a different representation of the technicolour gauge group than the part coupled directly to the electroweak sector. We choose it to be a massless Weyl fermion in the adjoint representation of the technicolour gauge group, that is a technigluino. The resulting theory has the same matter content as  $N_f$ -flavour super QCD but without the scalars; hence the name "split technicolour." We expect the critical number of flavours above which one enters the conformal window  $N_f^{\text{II}}$  to lie within the range

$$\frac{3}{2} < \frac{N_f^{\text{II}}}{N} < \frac{11}{2}. \quad (20)$$

The lower bound is the exact supersymmetric value for a non-perturbative conformal fixed point [32], while the upper bound is the one expected in the theory without a technigluino. The matter content of "split technicolour" lies between that of super QCD and the standard fundamental technicolour theory.

For two colours the number of (techni)flavours needed to be near the conformal window in the split case is at least three, while for three colours more than five flavours are required. These values are still larger than the ones for theories with fermions in the two-index symmetric representation. It is useful to remind the reader that in supersymmetric theories the critical number of flavours needed to enter the conformal window does not coincide with the critical number of flavours required to restore chiral symmetry. The scalars in supersymmetric theories play an important role from this point of view. We note that a split technicolour-like theory has been used recently in [35], to investigate the strong CP problem.

Split technicolour shares some features with theories of split supersymmetry recently advocated and studied in [33, 34] as possible extensions of the standard model. Clearly, we have introduced split technicolour—differently from split supersymmetry—to address the hierarchy problem. This is why we do not expect new scalars to appear at energy scales higher than the one of the electroweak theory.

#### IV. NEAR CONFORMAL SPECTRUM: REFINEMENTS.

An interesting question to ask is how the spectrum of light particles looks near the infrared fixed point. When these theories are used to dynamically break the electroweak symmetry this corresponds to asking, whether the composite Higgs for walking theories is light or heavy with respect to the electroweak symmetry breaking scale?

There is consensus that at least part of the hadronic spectrum becomes light near a conformal fixed point [25]. This might not be too surprising after all, since one expects the chiral symmetry breaking scale to vanish at the conformal fixed point. A more delicate issue is, if part of the spectrum and more specifically the chiral partner of the pion becomes light faster than the chiral symmetry breaking scale as we tune the number of flavours toward the fixed point value<sup>9</sup>.

We will use a simple analysis to elucidate why we expect to find a light composite scalar object, —light with respect to the chiral symmetry breaking scale—near the conformal fixed point. If, as function of the number of flavours, there is a smooth phase transition one applies Wilson's approach and investigates the correspond-

ing renormalisable scalar effective Lagrangian. In our analysis we are assuming that the rest of the composite spectrum of the theory decouples near this fixed point<sup>10</sup>. This is a standard assumption if a smooth transition occurs. Here the situation is different from the case of a thermal phase transition since fermions do not necessarily decouple at the four-dimensional fixed point.

For a generic number of space-time dimensions  $d$  one finds that the fixed point value of the  $\phi^4$  coupling constant  $g/4!$  of the scalar effective theory with an  $O(N_f)$  global symmetry is<sup>11</sup>,

$$g_* = \frac{48\pi^2}{N_f + 8}(4 - d). \quad (21)$$

Hence, in four dimensions, the effective fixed point coupling constant vanishes. This is consistent with the intuitive idea that near a conformal fixed point the presence of a non-zero coupling would lead to the generation of a mass term driving the theory away from conformality. Using the renormalisation group approach we compare the coupling constant at the chiral symmetry breaking scale  $\Sigma_0$ —that is  $g(\Sigma_0) = \bar{g}$ —with the same coupling at a much larger scale  $\Lambda$  which we identify with the scale above which the underlying coupling constant starts running again. We call the latter the bare coupling  $g$  and we have:

$$\begin{aligned} \bar{g} &\sim \frac{g}{1 + \frac{N_f + 8}{48\pi^2}g \left(\frac{\alpha_*}{\alpha_c} - 1\right)^{-1/2}} \sim \\ &\sim \frac{48\pi^2}{N_f + 8} \left(\frac{\alpha_*}{\alpha_c} - 1\right)^{1/2}. \end{aligned} \quad (22)$$

In the previous equation we used the following relation between the dynamically generated quark mass at zero momentum,  $\Sigma_0$ , and the scale  $\Lambda$  above which the underlying theory starts running again:

$$\ln\left(\frac{\Sigma_0}{\Lambda}\right) \sim \left(\frac{\alpha_*}{\alpha_c} - 1\right)^{-1/2}. \quad (23)$$

The function  $(\alpha_*/\alpha_c - 1)$  embodies the underlying dynamics, and, hence, is sensitive to the number of flavours, colours and the fermion representation with respect to the gauge group<sup>12</sup>. The physical mass of the scalar field in units of the vacuum expectation value of the scalar

<sup>9</sup> We thank T. Appelquist and K. Lane for pressing on this relevant point.

<sup>10</sup> Note that Chivukula [25] argued that the rest of the dynamics does not decouple near the same fixed point.

<sup>11</sup> This part is meant to be illustrative. If the reader is interested in more details on the fixed point analysis for theories with unitary flavour groups these can be found in [36].

<sup>12</sup> Nota Bene: The coefficient in front of  $(\alpha_*/\alpha_c - 1)$  depends on the underlying fermion representation with respect to the gauge group which dictates, ultimately, the global symmetry of the problem.

field is given by

$$\frac{M_\phi^2}{v^2} = 2\bar{g} \sim \left( \frac{\alpha_*}{\alpha_c} - 1 \right)^{1/2}, \quad (24)$$

with  $v = \langle \phi \rangle \sim \Sigma_0$ . Hence it vanishes near the transition faster than the vacuum expectation value, that is the chiral symmetry breaking scale. This supports the proposal that the composite Higgs can be lighter for walking technicolour theories than for ordinary technicolour theories away from conformality<sup>13</sup>.

There are some caveats related to the analysis above. If the transition is not smooth the present analysis is simply not applicable. This is the conclusion drawn, for example in [37] and in [38, 39]. Note that this uncertainty is present in every study of a phase transition via effective theories. Another issue is that for zero temperature phase transitions fermions are no longer screened near the fixed point and should be included in the analysis above. Besides, we could have also terms related to the underlying axial anomaly, depending on the number of flavours and the global flavour symmetry. However, if the phase transition happens near two flavours one expects the axial anomaly, which is related to a marginal operator, not to dominate the fixed point dynamics. Finally, we have also assumed valid the low energy effective description all the way to the scale  $\Lambda$ .

## V. SUMMARY

We have presented a comprehensive analysis of the phase diagram of non-supersymmetric vector-like and strongly coupled  $SU(N)$  gauge theories with matter in various representations of the gauge group. We have considered models with fermions in a single representation of the gauge group, but also the case of a combination of fermions in the fundamental with fermions in the adjoint representation. As physical application we considered the dynamical breaking of the electroweak symmetry via walking technicolour. We have then taken into account constraints from electroweak precision measurements and, thereby, reduced the number of theories viable for correctly describing the dynamical breaking. Still, we find that a considerable number of strongly coupled theories are still excellent candidates for breaking the electroweak symmetry dynamically.

<sup>13</sup> In our previous analysis [9] we suggested a dependence of the scalar mass on the number of flavours near the fixed point, using trace anomaly arguments. This result, however, did not take into account the scale dependence of the  $\phi^4$  coupling.

representation	Dynkin indices
singlet	(000...00)
fundamental (F)	(100...00)
antifundamental ( $\bar{F}$ )	(000...01)
adjoint (G)	(100...01)
$n$ -index symmetric ( $S_n$ )	(n00...00)
2-index antisymmetric ( $A_2$ )	(010...00)

TABLE IV: Examples for Dynkin indices for some common representations.

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## APPENDIX A: BASIC GROUP THEORY RELATIONS

The Dynkin indices label the highest weight of an irreducible representation and uniquely characterise the representations. The Dynkin indices for some of the most common representations are given in Table IV. For details on the concept of Dynkin indices see, for example [41, 42].

For a representation,  $R$ , with the Dynkin indices  $(a_1, a_2, \dots, a_{N-2}, a_{N-1})$  the quadratic Casimir operator reads [43]

$$C_2(R) = \sum_{m=1}^{N-1} [N(N-m)ma_m + m(N-m)a_m^2 + \sum_{n=0}^{m-1} 2n(N-m)a_n a_m] \quad (A1)$$

and the dimension of  $R$  is given by

$$d(R) = \prod_{p=1}^{N-1} \left\{ \frac{1}{p!} \prod_{q=p}^{N-1} \left[ \sum_{r=q-p+1}^p (1+a_r) \right] \right\}, \quad (A2)$$

which gives rise to the following structure

$$\begin{aligned} d(R) = & (1+a_1)(1+a_2)\dots(1+a_{N-1}) \times \\ & \times (1 + \frac{a_1+a_2}{2}) \dots (1 + \frac{a_{N-2}+a_{N-1}}{2}) \times \\ & \times (1 + \frac{a_1+a_2+a_3}{3}) \dots (1 + \frac{a_{N-3}+a_{N-2}+a_{N-1}}{3}) \times \\ & \times \dots \times \\ & \times (1 + \frac{a_1+\dots+a_{N-1}}{N-1}). \end{aligned} \quad (A3)$$

The Young tableau associated to a given Dynkin index  $(a_1, a_2, \dots, a_{N-2}, a_{N-1})$  is easily constructed. The

length of row  $i$  (that is the number of boxes per row) is given in terms of the Dynkin indices by the expression  $r_i = \sum_i^{N-1} a_i$ . The length of each column is indicated by  $c_k$ ;  $k$  can assume any positive integer value. Indicating the total number of boxes associated to a given Young tableau with  $b$  one has another compact expression for  $C_2(R)$ ,

$$C_2(R) = N \left[ bN + \sum_i r_i^2 - \sum_i c_i^2 - \frac{b^2}{N} \right], \quad (\text{A4})$$

and the sums run over each column and row.

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